On finiteness spaces and extensional presheaves over the Lawvere theory of polynomials

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Abstract

We define a faithful functor from a cartesian closed category of linearly topologized vector spaces over a field and generalized polynomial functions to the category of "extensional" presheaves over the Lawvere theory of polynomial functions, and show that, under some conditions on the field, this functor is full and preserves the cartesian closed structure.

Introduction

We introduced in [Ehr05] the notion of *finiteness space* for defining a new denotational model[1] of classical linear logic [Gir87]. Finiteness spaces are "discrete" objects presenting similarities with coherence [Gir87] or hypercoherence [Ehr93] spaces. There are many differences however.

In particular, whereas morphisms between coherence or hypercoherence spaces are defined in a domain-theoretic way and act on the sets of *cliques* (partially ordered under inclusion) of the spaces[2], morphisms of finiteness spaces are defined once a field $\mathbf{k}$ has been chosen. Indeed, one can then associate with each finiteness space X a *linearly topologized vector space*[3] $\mathbf{k}\langle X\rangle$, and a morphism from the finiteness space X to the finiteness space Y is simply a linear and continuous map between the associated Lefschetz spaces $\mathbf{k}\langle X\rangle$ and $\mathbf{k}\langle Y\rangle$.

The advantage of finiteness spaces, as compared to arbitrary Lefschetz spaces, is that they lead quite easily to a model of classical linear logic[4], and that the space constructions required for this purpose (tensor product, topological dual, linear function space, direct product and coproduct and "exponentials") can be done without any mention to a particular choice of field, while admitting a standard algebraic interpretation once a field is given. For instance, the tensor product of finiteness spaces can be shown to classify the bilinear *hypocontinuous*[5] maps. On the other hand, finiteness spaces are rather rigid objects, and are not obviously stable under many important operations such as taking subspaces or quotients.

The exponential "!" on finiteness spaces is an operation which, given any field of scalars, induces a monoidal comonad on the category of finiteness spaces and continuous linear maps, and the Kleisli category of this comonad is cartesian closed (this is the standard situation of a model of linear logic, see [Bie95]).

[1] That is, a model where formulae are interpreted as spaces, and proofs as morphisms.
[2] Coherence and hypercoherence spaces are some kinds of graphs.
[3] A notion introduced by Lefschetz in [Lef42], simpler than the usual notion of locally convex topological vector space. We call these spaces simply Lefschetz spaces in the sequel. Note that in this setting, the field $\mathbf{k}$ is considered with the discrete topology.
[4] In particular, any such space is canonically isomorphic to its topological bidual, topological duals being equipped with a suitable standard topology: the topology of uniform convergence on all *linearly compact* subspaces.
[5] A notion of continuity stronger than separate continuity and weaker than continuity wrt. the product topology.

The morphisms in this Kleisli category can also be seen as functions between the Lefschetz spaces associated with finiteness spaces, and we are mainly interested here in characterizing these particular functions.

We provide first a rather straightforward characterization. Since we have a good notion of multilinear maps between Lefschetz spaces (the hypocontinuous ones), we also have a standard notion of polynomial functions between such spaces (given a finite family of hypocontinuous multilinear maps $f_n : E^n \to F$, the map $x \mapsto \sum_n f_n(x, \dots, x)$ is polynomial). We show that morphisms in the Kleisli category are the elements of the completion of this space of polynomial functions, equipped with the topology of uniform convergence on all *linearly compact*[6] subspaces.

We provide then a less direct characterization. Up to isomorphism, the finite dimensional Lefschetz spaces are simply the $\mathbf{k}^n$s, equipped with the discrete topology, and they are finiteness spaces. The Kleisli morphisms between such spaces are the (tuples of) polynomial functions. So the Kleisli category is a cartesian closed category which contains the category of finite dimensional spaces and polynomial maps[7] as a full subcategory. But there is another, standard, cartesian closed extension of this category, namely the topos of presheaves over it. We show that the Yoneda embedding extends to a *full* and faithful functor from the Kleisli category to the category of presheaves over the category of polynomials, and that this functor preserves the cartesian closed structure. The presheaves in the range of this functor are of a particular kind, they are *extensional presheaves* (see [RS99, Str05]).

For the time being, we are able to prove this result only under a restrictive condition on the field: it has to admit an absolute value for which it is non-discrete and complete. This is the case of the fields of real or complex numbers, and of the fields of p-adic numbers, but this is not the case for the field of rational numbers, or, even worse, for finite fields. The reason for this restriction is that we use at some point the Baire categoricity theorem and the fact that a polynomial of n variables with coefficients in $\mathbf{k}$ is 0 as soon as it vanishes on a set of the shape I^n, where I is an infinite subset of $\mathbf{k}$.

It is not the first time that one observes that cartesian closed categories induced by models of linear logic are cartesian closed subcategories of extensional presheaf categories, see [CE94] where a similar phenomenon is observed for strongly stable functions on hypercoherences. The same can be shown for stable functions on coherence spaces.

Extensional presheaves have been used by various authors for extending standard "first order" concepts to higher type functions. For instance, Hofmann extended polytime computation to higher types in [Hof97]. This approach is described from a more general perspective in [RS99].

Similar approaches, adopted by Chen [Che77] and then independently by Souriau [Sou81] gave rise to the concept of *diffeological space*, a very nice categorical setting generalizing smooth manifolds, where a lot of natural constructions (such as function spaces) become available. In their work, smoothness is extended to higher types.

1 Basic notions

Notations. We use the symbol $\mathbf{N}$ to denote the set of *positive* integers $\{1, 2, \dots\}$. Let $\mathbf{k}$ be an infinite field.

If E is a $\mathbf{k}$-vector space, an affine subset G of E is a subset of E which is stable under barycentric linear combinations (that is, linear combinations where coefficients have sum equal to 1). Such an affine subset has a *direction* $\mathsf{dir}\, G$, which is the linear subspace $-x + G$, where x is an arbitrary element of G. An affine subset is a linear subspace iff it contains 0. Equivalently, an affine subset is a set of the shape $x + D$ where $x \in E$ and D is a linear subspace of E (and then of course $\mathsf{dir}(x + D) = D$).

When we say that a property holds for *almost all* the elements of a collection, we mean that it holds for all, but maybe a finite number of elements of that collection.

Given a function f from a set S to a set M which contains a 0 element (in the sequel, M will be a field or the set of natural numbers), the *support* of f is the set of all the elements of S which are not mapped to 0 by f. This set is denoted as $|f|$. A *finite multiset* of elements of S is a map from S to $\mathbb{N}$ whose support is

[6] A notion of compactness adapted to the linearly topologized setting.

[7] Up to the choice of one canonical representative for each finite dimension, this is a Lawvere theory [Law63].

finite. We denote by $[s_1, \ldots, s_n]$ the finite multiset which maps $s \in S$ to the number of times s appears in the list $s_1, \ldots, s_n$ of elements of S.

1.1 The Lawvere theory of polynomials.

We denote by $\mathbf{Pol_k}$ the following small category: the objects of $\mathbf{Pol_k}$ are the natural numbers, and, given $p, q \in \mathbb{N}$, a morphism from p to q in $\mathbf{Pol_k}$ is a tuple $(f_1, \ldots, f_q)$ of elements of $\mathbf{k}[\xi_1, \ldots, \xi_p]$. This category is cartesian, the cartesian product being given by the sum of natural numbers, and the terminal object being 0. It is a typical example of a *Lawvere theory* [Law63].

A morphism $f \in \mathbf{Pol_k}(p, q)$ gives rise to a function from $\mathbf{k}^p$ to $\mathbf{k}^q$. It is well known that, since $\mathbf{k}$ is infinite, this correspondence is injective. In other terms, $\mathbf{Pol_k}$ has enough points.

From now on, we freely consider the morphisms of $\mathbf{Pol_k}$ as functions.

1.2 Lefschetz spaces and finiteness spaces

In this section, $\mathbf{k}$ is equipped with the discrete topology. We refer to [Köt69] for the general theory of Lefschetz spaces. We give first a short introduction to this theory, defining only the concepts which are useful for the present paper.

1.2.1 General Lefschetz spaces

A linearly topologized $\mathbf{k}$-vector space, called $\mathbf{k}$-*Lefschetz space* in the present paper, is a $\mathbf{k}$-vector space E equipped with a topology λ_E which has the following property: there exists a filter base $\mathcal{V}$ of linear subspaces of E such that a subset U of E is λ_E-open iff for any $x \in U$ there exists $V \in \mathcal{V}$ such that $x + V \subseteq U$. Such a system $\mathcal{V}$ will be called a *fundamental system* at 0 for the topology λ_E. Then, E is Hausdorff iff $\bigcap \mathcal{V} = \{0\}$; we shall always assume that this is the case (this assumption is part of our definition of a Lefschetz space).

Addition and scalar multiplication are continuous maps ($E \times E$ and $\mathbf{k} \times E$, respectively, being endowed with the product topology). Given E and F two Lefschetz spaces, a linear morphism from E to F is a continuous linear function from E to F. We denote by $\mathcal{L}(E, F)$ the set of these morphisms, which is a $\mathbf{k}$-vector space.

Let U be a linear open subspace of E and let $x \in E \setminus U$. Then if there were $y \in U \cap (x + U)$, we would have $y - x \in U$ as well as $y \in U$ and hence, since $x = y - (y - x) \in U$, a contradiction. Therefore, any linear open subspace is also closed. This shows by the way that E is a totally disconnected topological space.

If F is a linear subspace of E (with topology λ_E), equipped with the induced topology, then F is also a Lefschetz space, and if $\mathcal{V}$ is a fundamental system at 0 for λ_E, then $\mathcal{W} = \{U \cap F \mid U \in \mathcal{V}\}$ is a fundamental system at 0 for the induced topology on F.

We recall that a *net* in E is a family $(x(\alpha))_{\alpha \in \Gamma}$ of elements of E indexed by a directed poset Γ. Such a net converges to $x \in E$ if, for any neighborhood V of 0, there exists $\alpha \in \Gamma$ such that, for any $\beta \in \Gamma$ with $\beta \geq \alpha$, one has $x - x(\alpha) \in V$. If the net converges, it converges to a unique value because E is Hausdorff.

A subspace K of E is *linearly compact* if, for any filter base $\mathcal{G}$ of closed affine subspaces of E, if $K \cap G \neq \emptyset$ for each $G \in \mathcal{G}$, one has $K \cap \bigcap \mathcal{G} \neq \emptyset$. Obviously, any closed subspace of a linearly compact subspace is also linearly compact.

Observe that if K is linearly compact, then K is closed. Indeed, let $x \in E$ and let $\mathcal{U}$ be a fundamental system at 0 for λ_E. Assume that $(x + U) \cap K \neq \emptyset$ for each $U \in \mathcal{U}$. Since any open linear subspace is also closed, all the affine spaces $x + U$ are closed, and hence $\mathcal{G} = (x + U)_{U \in \mathcal{U}}$ is a filter base of closed affine spaces which intersect K. Therefore $K \cap \bigcap_{U \in \mathcal{U}} (x + U) \neq \emptyset$, that is, $x \in K$.

Of course, the direct image of a linearly compact subspace by a linear and continuous map is also linearly compact.

Complete Lefschetz spaces, completion. As it is usual for topological vector spaces, a Lefschetz space E is a uniform space: for each linear neighborhood U or 0 in E, we can define a basic entourage $\mathsf{ent}(U) = \{(x, y) \in E \times E \mid x - y \in U\}$. These basic entourages generate a uniform structure on E (all the subsets of $E \times E$ which contain some $\mathsf{ent}(U)$). One says that E is complete if it is complete with respect to this uniform structure.

Similarly, the completion of a Lefschetz space E is defined as the completion of this canonically associated uniform space. More precisely, a uniform space F is a completion of E if F is complete and contains E as a dense subspace. This determines F up to unique homeomorphism, so we denote F as $\tilde{E}$. This space $\tilde{E}$ always exists and has an unique $\mathbf{k}$-vector space structure which extends that of E and turns it into a $\mathbf{k}$-Lefschetz space.

We spell out in the present setting the notion of Cauchy net and the definition of completeness.

Let Γ be a directed partially ordered set. A net $(x(\alpha))_{\alpha\in\Gamma}$ is Cauchy if, for any neighborhood V or 0, there exists $\alpha \in \Gamma$ such that, for any $\beta, \beta' \in \Gamma$ with $\beta, \beta' \geq \alpha$, one has $x(\beta) - x(\beta') \in V$. Saying that E is complete is equivalent to saying that any Cauchy net converges in E.

Observe that, in a complete Lefschetz space, a series converges iff its general term $x(n)$ goes to 0 as n goes to infinity.

Space constructions. Concerning the constructions which can be performed on general Lefschetz spaces, we mention here only function spaces: if E and F are Lefschetz spaces, we already defined the $\mathbf{k}$-vector space $\mathcal{L}(E, F)$ of linear continuous maps from E to F. We consider always this space as endowed with the linear topology whose fundamental system of neighborhoods of 0 is given by the subspaces

$$\mathcal{W}(K, V) = \{f \in \mathcal{L}(\mathbf{k}\langle X\rangle, \mathbf{k}\langle Y\rangle) \mid f(K) \subseteq V\}\,.$$

where $K \subseteq E$ is an arbitrary linearly compact subspace and $V \subseteq F$ is an arbitrary linear neighborhood of 0. This space is a $\mathbf{k}$-Lefschetz space as easily checked. In particular, $E' = \mathcal{L}(E, \mathbf{k})$ is the topological dual of E.

The collection of all $\mathcal{W}(K, V)$ is a filter base because, if $K, L \subseteq E$ are linearly compact, then so is $K + L$ as the range of the canonical map $K \oplus L \to K + L$ which is linear and continuous and because $K \oplus L$ is linearly compact as the product of two linearly compact spaces[8].

1.2.2 Finiteness spaces

In this section, we essentially summarize the definitions and results from [Ehr05] which are useful for our purpose.

Let I be a set. Given a subset $\mathcal{I}$ of $\mathcal{P}(I)$, let $\mathcal{I}^\perp$ be the collection of all the elements of $\mathcal{P}(I)$ which have a finite intersection with all the elements of $\mathcal{I}$:

$$\mathcal{I}^\perp = \{u' \subseteq I \mid \forall u \in \mathcal{I} \;\; u \cap u' \text{ is finite}\}\,.$$

A *finiteness space* is a pair $X = (|X|, \mathsf{F}(X))$ where $|X|$ is an at most countable set and $\mathsf{F}(X) \subseteq \mathcal{P}(|X|)$ is such that $\mathsf{F}(X) = \mathsf{F}(X)^{\perp\perp}$ (that is, actually, $\mathsf{F}(X)^{\perp\perp} \subseteq \mathsf{F}(X)$ since the converse inclusion always holds). One observes immediately that $\mathsf{F}(X)$ contains all finite sets, is closed under finite unions and that, if $u \in \mathsf{F}(X)$, any subset of u belongs to $\mathsf{F}(X)$.

Let X be a finiteness space, we define a $\mathbf{k}$-vector space $\mathbf{k}\langle X\rangle$ as follows: the elements of this space are the elements of $\mathbf{k}^{|X|}$ whose support belongs to $\mathsf{F}(X)$. Given $u' \in \mathsf{F}(X)^\perp$, let $V_{u'} \subseteq \mathbf{k}\langle X\rangle$ be the set of all $x \in \mathbf{k}\langle X\rangle$ such that $x_a = 0$ for all $a \in u'$. Observe that $V_{u'} \cap V_{v'} = V_{u'\cup v'}$ and equip $\mathbf{k}\langle X\rangle$ with the topology λ_X for which a subset U of $\mathbf{k}\langle X\rangle$ is open iff for any $x \in U$, there is $u' \in \mathsf{F}(X)^\perp$ such that $x + V_{u'} \subseteq U$. Then $(\mathbf{k}\langle X\rangle, \lambda_X)$ is clearly a Lefschetz space whose topology admits $\{V_{u'} \mid u' \in \mathsf{F}(X)^\perp\}$ as fundamental system at 0. Moreover this space is always complete[9].

The space $\mathbf{k}^{(|X|)}$ is always a linear subspace of $\mathbf{k}\langle X\rangle$, and lies densely in that space. Indeed, let $x \in \mathbf{k}\langle X\rangle$ and let $u' \in \mathsf{F}(X)^\perp$. We know that $u_0 = |x| \cap u'$ is finite, so let $y \in \mathbf{k}^{(|X|)}$ be the vector which vanishes outside u_0 and takes the same values as x on u_0. Then $x - y \in V_{u'}$.

If $|X|$ is finite, then $\mathsf{F}(X) = \mathcal{P}(|X|)$ and $\mathbf{k}\langle X\rangle = \mathbf{k}^{|X|}$, with the discrete topology. When $|X|$ is infinite, there are many possibilities for $\mathsf{F}(X)$, between two extreme situations:

[8] Any product of linearly compact spaces is linearly compact.

[9] This is another virtue of the inclusion $\mathsf{F}(X)^{\perp\perp} \subseteq \mathsf{F}(X)$.

- the situation where $\mathsf{F}(X)$ is the set of all finite subsets of $|X|$, and then $\mathbf{k}\langle X\rangle = \mathbf{k}^{(|X|)}$ is the set of all the elements of $\mathbf{k}^{|X|}$ which vanish at almost all element of $|X|$, and this space is equipped with the discrete topology
- and the situation where $\mathsf{F}(X) = \mathcal{P}(|X|)$ and then $\mathbf{k}\langle X\rangle = \mathbf{k}^{|X|}$, equipped with the product topology (remember that $\mathbf{k}$ has the discrete topology).

Let us come back to the general situation. The following result was stated but not proved in [Ehr05].

Theorem 1 *Let K be a linear subspace of $\mathbf{k}\langle X\rangle$. Then K is linearly compact iff K is closed and $\bigcup\{|x| \mid x \in K\} \in \mathsf{F}(X)$.*

Proof. Assume first that K is closed and that $u \in \mathsf{F}(X)$, where $u = |K| = \bigcup\{|x| \mid x \in K\}$ and let us show that K is linearly compact. For that, it is enough to show that the subspace L of all the elements x of $\mathbf{k}\langle X\rangle$ such that $|x| \subseteq u$ is linearly compact. But L is closed (as the intersection of the closed sets $\{x \mid x_a = 0\}$ for $a \in |X| \setminus u$) and hence it suffices to show that L is a linearly compact Lefschetz space; but L is linearly homeomorphic to $\mathbf{k}^u$ with the product topology since $u \in \mathsf{F}(X)$. We conclude by the fact that the product of any family of linearly compact spaces is linearly compact (of course, $\mathbf{k}$ itself is linearly compact), see [Köt69].

Conversely, assume that K is linearly compact, let $u = |K|$ and let us show that $u \in \mathsf{F}(X)$. If u is finite, there is nothing to say, so assume that u is infinite and let $a_1, a_2, \dots$ be a repetition-free enumeration of this set. Choose $x(1) \in K$ such that $x(1)_{a_1} = 1$. Consider the linear map $\rho : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle$ which maps y to $y_{a_1} x(1)$. This projection is continuous because the coordinate linear form $y \mapsto y_{a_1}$ is and let $K_1 = K \cap \ker \rho = \{x \in K \mid x_{a_1} = 0\}$. This space K_1 is a closed subspace of K and we have $K = \mathbf{k}x(1) \oplus K_1$. By construction, $|K_1| \subseteq \{a_2, a_3, \dots\}$. If $K_1 \neq 0$, we can reiterate this process. There are two possibilities:

- either we find a finite sequence of vectors $x(1), \dots, x(N)$ with $K = \mathbf{k}x(1) \oplus \cdots \oplus \mathbf{k}x(N)$ and we conclude easily, because then $u \subseteq |x_1| \cup \cdots \cup |x(n)| \in \mathsf{F}(X)$,
- or we define an infinite sequence of vectors $x(i) \in K \setminus \{0\}$ and of closed subspaces K_i of K, with the property that $K_i = \mathbf{k}x(i+1) \oplus K_{i+1}$ and the least j such that $a_j \in |x_i|$ is strictly less than the least j such that $a_j \in |x(i+1)|$.

Assume that we are in the second situation. For any $x \in K$, we can define a sequence $\lambda_1, \lambda_2 \dots$ of elements of $\mathbf{k}$ such that, for each n, $x - \lambda_1 x(1) - \cdots - \lambda_n x(n) \in K_n$. In this way, we define an injective linear map φ from K to $\mathbf{k}^{\mathbf{N}}$. Conversely, if $\lambda \in \mathbf{k}^{\mathbf{N}}$, set $G_n = \lambda_1 x(1) + \cdots + \lambda_n x(n) + K_n$. Then G_n is a closed affine subspace of K (since K_n is closed). Moreover, $G_{n+1} \subset G_n$. Hence, since K is linearly compact, $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$ and let x be an (actually, the unique) element of this set. Then it is clear that $\varphi(x) = \lambda$ and we have shown that φ is a linear isomorphism between K and $\mathbf{k}^{\mathbf{N}}$.

Observe that we have $u = \bigcup_{i \in \mathbf{N}} |x(i)|$.

Let $u' \in \mathsf{F}(X)^{\perp}$ and assume towards a contradiction that $u \cap u'$ is infinite (remember that we must show that $u \in \mathsf{F}(X) = \mathsf{F}(X)^{\perp\perp}$). Let $j(1)$ be the least j such that $a_j \in u \cap u'$. Let $i(1)$ be the least i such that $a_{j(1)} \in |(x(i))|$. We know that $|x(i(1))| \cap u'$ is finite, and hence there is some $n \in \mathbf{N}$ such that $|x(i)| \cap (|x(i(1))| \cap u') = \emptyset$ for any $i \geq n$ (this holds by definition of the $x(i)$s). Moreover, $u' \cap (\bigcup_{i<n} |x(i)|)$ is finite, and hence we can find $j(2) > j(1)$ with $a_{j(2)} \in u' \cap u$ and $a_{j(2)} \notin u' \cap (\bigcup_{i<n} |x(i)|)$. Let $i(2)$ be the least i such that $a_{j(2)} \in |(x(i))|$ (we must have therefore $i \geq n$). By construction, we have $(|x(i(1))| \cap u') \cap (|x(i(2))| \cap u') = \emptyset$. Continuing this process, we build two strictly monotone sequences $(j(l))_{l \in \mathbf{N}}$ and $(i(l))_{l \in \mathbf{N}}$ with the following properties

- $a_{j(l)} \in u \cap u' \cap |x(i(l))|$ for each l
- and $(|x(i(l))| \cap u') \cap (|x(i(l'))| \cap u') = \emptyset$ as soon as $l \neq l'$.

Let $\lambda \in \mathbf{k}^{\mathbf{N}}$ be the characteristic function of the sequence $(i(l))_{l \in \mathbf{N}}$ (that is, $\lambda_i = 1$ if i is some i_l, and $\lambda_i = 0$ otherwise). Then clearly, for each l, we have $a_{j(l)} \in |\varphi^{-1}(\lambda)|$ and this is impossible since $u' \cap |\varphi^{-1}(\lambda)|$ must be finite. □

Tensor product and linear function space. Given two finiteness spaces X and Y, one defines $X \otimes Y$ as follows: $|X \otimes Y| = |X| \times |Y|$, and a subset w of this web belongs to $\mathsf{F}(X \otimes Y)$ iff its first projection belongs to $\mathsf{F}(X)$ and its second projection belongs to $\mathsf{F}(Y)$. With this definition, one can check that indeed $\mathsf{F}(X \otimes Y) = \mathsf{F}(X \otimes Y)^{\perp\perp}$. Then consider the finiteness space $X \multimap Y = (X \otimes Y^{\perp})^{\perp}$. Given $M \in \mathbf{k}\langle X \multimap Y\rangle$, one shows easily that the map $f : \mathbf{k}\langle X\rangle \to \mathbf{k}^{|Y|}$ given by

$$f(x)_b = \sum_{a \in |X|} M_{a,b} x_a \tag{1}$$

is well defined (all these sums have only a finite number of non-zero terms, because $|x| \in \mathsf{F}(X)$ and $|M| \in \mathsf{F}(X \multimap Y)$), takes its values in $\mathbf{k}\langle Y\rangle$, and is a continuous linear map from $\mathbf{k}\langle X\rangle$ to $\mathbf{k}\langle Y\rangle$. One can also write $f(x) = Mx$ since Formula (1) above corresponds to the usual application of a matrix to a vector.

Conversely, any $f \in \mathcal{L}(\mathbf{k}\langle X\rangle, \mathbf{k}\langle Y\rangle)$ is definable by such a matrix $M \in \mathbf{k}\langle X \multimap Y\rangle$, and in that way, we have exhibited a linear homeomorphism between the $\mathbf{k}$-Lefschetz spaces $\mathbf{k}\langle X \multimap Y\rangle$ and $\mathcal{L}(\mathbf{k}\langle X\rangle, \mathbf{k}\langle Y\rangle)$. This is just a generalization of the usual correspondence between linear maps and matrices, once bases have been chosen for the source and target vector spaces; here the sets $|X|$ and $|Y|$ are the corresponding bases.

This shows in particular that the topological bidual of $\mathbf{k}\langle X\rangle$ is canonically isomorphic to $\mathbf{k}\langle X\rangle$: the category of finiteness spaces and linear and continuous functions between the associated Lefschetz spaces, equipped with the tensor product described above, is $*$-autonomous [Bar79, Bie95]. The unit of the tensor product is the finiteness space 1 whose web is a singleton, so that $\mathbf{k}\langle 1\rangle = \mathbf{k}$, and this is also the dualizing object.

Hypocontinuous bilinear maps. Until now we have dealt only with continuous linear maps. The presence of a tensor product suggests of course that some bilinear maps are also of interest. Indeed, the map $\tau : \mathbf{k}\langle X\rangle \times \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle X \otimes Y\rangle$ which maps (x, y) to $x \otimes y$ defined by $(x \otimes y)_{a,b} = x_a y_b$ is bilinear, and separately continuous, but not continuous in general: it is only *hypocontinuous*.

Given three Lefschetz space E, F and G, we shall say that a bilinear map $h : E \times F \to G$ is hypocontinuous if, given $K \subseteq E$ linearly compact and $W \subseteq G$ linear open, there exists $V \subseteq F$ linear open such that $h(K \times V) \subseteq W$, and similarly swapping the roles of E and F. (This is an adaptation of the standard concept: in the standard locally convex setting, “compact” is replaced by “bounded”). One defines similarly the notion of hypocontinuous n-linear maps $E_1 \times \cdots \times E_n \to F$: for any $i = 1, \ldots, n$ and any linearly compact subspaces $K_j \subseteq E_j$ $(j \neq i)$ and any linear neighborhood V of F, there is a linear neighborhood U of 0 in E_i such that $f(K_1 \times \cdots \times K_{i-1} \times U \times K_{i+1} \times \cdots \times K_n) \subseteq V$.

One can show that τ has the corresponding universal property: any hypocontinuous bilinear map $h : \mathbf{k}\langle X\rangle \times \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle Z\rangle$ factors through τ followed by a continuous linear map $\mathbf{k}\langle X \otimes Y\rangle \to \mathbf{k}\langle Z\rangle$. This is the case of the typical bilinear map, namely the evaluation map $\mathbf{k}\langle X\rangle \times \mathbf{k}\langle X^{\perp}\rangle \to \mathbf{k}$ – it maps x (vector) and x' (covector, or continuous linear form), to $\langle x, x'\rangle = \sum_{a \in |X|} x_a x'_a$.

Local linear compactness. One can show that the bilinear map $(x, x') \mapsto \langle x, x'\rangle$ is continuous (for the product topology) iff the topology of $\mathbf{k}\langle X\rangle$ is locally linearly compact (that is, generated by a filter base of linearly compact subspaces), a condition which is equivalent to saying that $|X| = u \cup u'$ for some $u \in \mathsf{F}(X)$ and $u' \in \mathsf{F}(X)^{\perp}$. This property is stable under many constructions, but not under the exponential constructions, which are needed for defining a cartesian closed category. This explains why the morphisms in this cartesian closed category *will not be continuous*, at least wrt. the topology considered so far on Lefschetz spaces generated by finiteness spaces.

Cartesian product. The cartesian product (which is also the direct sum) of two finiteness spaces X and Y is $X \mathbin{\&} Y$ with $|X \mathbin{\&} Y| = |X| + |Y|$ (disjoint union – we assume that $|X| \cap |Y| = \emptyset$ for simplicity) and $w \in \mathsf{F}(X \mathbin{\&} Y)$ if $w \cap |X| \in \mathsf{F}(X)$ and $w \cap |Y| \in \mathsf{F}(Y)$. It is clear that $\mathbf{k}\langle X \mathbin{\&} Y\rangle = \mathbf{k}\langle X\rangle \times \mathbf{k}\langle Y\rangle = \mathbf{k}\langle X\rangle \oplus \mathbf{k}\langle Y\rangle$. Projections and pairing of maps are defined in the usual way.

1.3 Exponentials and the cartesian closed category of finiteness spaces

We define $!X$ as follows: $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ (the set of all finite multisets of elements of $|X|$), and a subset M of $|!X|$ belongs to $\mathsf{F}(!X)$ if $|M| = \cup\{|m| \mid m \in M\} \in \mathsf{F}(X)$; one can prove indeed that with this definition, $\mathsf{F}(!X) = \mathsf{F}(!X)^{\perp\perp}$. Given $x \in \mathbf{k}\langle X\rangle$ and $m \in |!X|$, we define $x^m \in \mathbf{k}$ by $x^m = \prod_{a\in|X|} x_a^{m(a)}$, this product being actually finite, and different from 0 iff $|m| \subseteq |x|$. For that reason, setting $x^! = (x^m)_{m\in|!X|}$, we have $x^! \in \mathbf{k}\langle !X\rangle$. Of course (just as for the map τ in the case of the tensor product), this map $x \mapsto x^!$ is neither linear nor continuous in general. Given a matrix $M \in \mathbf{k}\langle !X \multimap Y\rangle$, we can define a map $\widehat{M} : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$ given by $\widehat{M}(x)_b = \sum_{m\in|!X|} M_{m,b} x^m$, that is $\widehat{M}(x) = Mx^!$, applying the matrix M to the vector $x^!$.

It is interesting to observe that, in spite of its apparent simplicity, a few iterations of this exponential construction and of topological duality lead, starting from the simplest space (0, the 0-dimensional space), to finiteness spaces which are complicated in the sense that their associated Lefschetz spaces are not even metrizable. This phenomenon has certainly a logical and computational meaning, which is not very clear yet.

One can turn the operation $X \mapsto !X$ into a functor as follows. Let $M \in \mathbf{k}\langle X \multimap Y\rangle$, we define $!M \in \mathbf{k}^{|!X\multimap !Y|}$ by the following formula:

$$(!M)_{m,p} = \sum_{r\in L(m,p)} \begin{bmatrix} p \\ r \end{bmatrix} M^r$$

where $L(m,p)$ is the set of all $r \in \mathcal{M}_{\mathrm{fin}}(|X| \times |Y|)$ such that, for any $a \in |X|$, one has $\sum_{b\in|Y|} r(a,b) = m(a)$ and, for any $b \in |Y|$, one has $\sum_{a\in|X|} r(a,b) = p(b)$, and $\left[\begin{smallmatrix} p \\ r \end{smallmatrix}\right]$ is the multinomial coefficient (a natural number), given by $\left[\begin{smallmatrix} p \\ r \end{smallmatrix}\right] = \prod_b p(b)!/\prod_{a,b} r(a,b)!$. One checks easily that $!M$ is well defined by this formula, and belongs to $\mathsf{F}(!X \multimap !Y)$; one can check indeed that

$$|!M| = \{([a_1,\dots,a_n],[b_1,\dots,b_n]) \mid \forall i\,(a_i,b_i) \in |M|\}\,.$$

This functor is a comonad with the following structure: $\mathrm{d}^X \in \mathbf{k}\langle !X \multimap X\rangle$ given by $\mathrm{d}^X_{m,a} = \delta_{m,[a]}$ and $\mathrm{p}^X \in \mathbf{k}\langle !X \multimap !!X\rangle$ given by $\mathrm{p}^X_{m,[m_1,\dots,m_n]} = \delta_{m,m_1+\cdots+m_n}$ (δ stands for the Kronecker symbol).

The Kleisli category of this comonad can be shown to be cartesian closed. In this category, where a morphism from X to Y is an element of $\mathbf{k}\langle !X \multimap Y\rangle$, one can check that identity (defined as d^X) satisfies $\widehat{\mathrm{Id}}(x) = x$ and that composition (defined using the action of the functor "!" on morphism and p^X) satisfies $\widehat{P\circ M} = \widehat{P} \circ \widehat{M}$. Moreover, since $\mathbf{k}$ is infinite, the mapping $M \mapsto \widehat{M}$ is injective, so that we can consider the morphisms of this category as *functions*: a morphism from X to Y is a function $f : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$ such that there exists $M \in \mathbf{k}\langle !X \multimap Y\rangle$ satisfying $f = \widehat{M}$. We denote by $\mathbf{Fin}^!_{\mathbf{k}}$ this category.

The cartesian product in this Kleisli category is the same as the cartesian product defined previously, in the underlying "linear category" of finiteness spaces and linear and continuous maps. This is due to the monoidal closedness of that category, and to the isomorphism $!(X \mathbin{\&} Y) \simeq !X \otimes !Y$.

The function space of X and Y is $X \Rightarrow Y = !X \multimap Y$. By definition, $\mathbf{k}\langle X \Rightarrow Y\rangle$ is linearly isomorphic to the vector space $\mathbf{Fin}^!_{\mathbf{k}}(X,Y)$. Considering the elements of this vector space as functions, the evaluation map $\mathrm{ev} \in \mathbf{Fin}^!_{\mathbf{k}}((X \Rightarrow Y) \mathbin{\&} X, Y)$ is defined in the standard way: $\mathrm{ev}(f,x) = f(x)$. If $f \in \mathbf{Fin}^!_{\mathbf{k}}(Z \mathbin{\&} X, Y)$ is seen as a function $f : \mathbf{k}\langle Z\rangle \times \mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$, the "curryfication" of f is a morphism $g \in \mathbf{Fin}^!_{\mathbf{k}}(Z, X \Rightarrow Y)$ which maps $z \in \mathbf{k}\langle Z\rangle$ to the function $x \mapsto f(z,x)$, just as in the category of sets and functions.

An intrinsic presentation of function spaces. Let E and F be Lefschetz spaces. Let us say that a function $f : E \to F$ is *polynomial*[10] if there is $n \in \mathbf{N}$ and hypocontinuous i-linear maps $f_i : E^i \to F$ (for $i = 0,\dots,n$) such that

$$f(x) = f_0 + f_1(x) + \cdots + f_n(x,\dots,x)\,.$$

A polynomial map f of the form $f(x) = f_n(x,\dots,x)$, where f_n is an n-linear hypocontinuous function, is said to be homogeneous of degree n (it implies of course $f(tx) = t^n f(x)$).

[10]This kind of definition is completely standard: it is in that way that one defines e.g. polynomial functions between Banach spaces.

Let $\mathbf{Pol}_{\mathbf{k}}(E,F)$ be the $\mathbf{k}$-vector space of polynomial functions from E to F. This space can be endowed with the linear topology of uniform convergence on all linearly compact subspaces, which admits the following generating filter base of open neighborhoods of 0: basic opens are $\mathcal{W}(K,V)=\{f\in\mathbf{Pol}_{\mathbf{k}}(E,F)\mid f(K)\subseteq V\}$, where $K\subseteq E$ is linearly compact[11] and V is an open subspace of F. Let $\mathcal{A}(E,F)$ be the completion of that Lefschetz space.

Theorem 2 *For any finiteness spaces X and Y, the Lefschetz space $\mathbf{k}\langle X\Rightarrow Y\rangle$ is linearly homeomorphic to $\mathcal{A}(\mathbf{k}\langle X\rangle,\mathbf{k}\langle Y\rangle)$.*

Proof. Any polynomial map from $\mathbf{k}\langle X\rangle$ to $\mathbf{k}\langle Y\rangle$ is an element of $\mathbf{Fin}^!_{\mathbf{k}}(X,Y)$ as easily checked; we have an inclusion $\mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle,\mathbf{k}\langle Y\rangle)\subseteq\mathbf{k}\langle X\Rightarrow Y\rangle$. Conversely, let $(m,b)\in|X\Rightarrow Y|$ with $m=[a_1,\ldots,a_n]$. The map $f:\mathbf{k}\langle X\rangle^n\to\mathbf{k}$ which maps $(x(1),\ldots,x(n))$ to the product $x(1)_{a_1}\ldots x(n)_{a_n}$ is multilinear and hypocontinuous. Hence the same holds for the map $x\mapsto f(x)e_b$ from $\mathbf{k}\langle X\rangle^n$ to $\mathbf{k}\langle Y\rangle$. Therefore we have $\mathbf{k}^{(|X\Rightarrow Y|)}\subseteq\mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle,\mathbf{k}\langle Y\rangle)$. Hence $\mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle,\mathbf{k}\langle Y\rangle)$ is a dense subspace of $\mathbf{k}\langle X\Rightarrow Y\rangle$. To show that $\mathbf{k}\langle X\Rightarrow Y\rangle$ is the completion of $\mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle,\mathbf{k}\langle Y\rangle)$ it suffices to show that the above defined linear topology on that space (uniform convergence on all linearly compact subspaces) is the restriction of the topology of $\mathbf{k}\langle X\Rightarrow Y\rangle$.

Let $K\subseteq\mathbf{k}\langle X\rangle$ be linearly compact and let $V\subseteq\mathbf{k}\langle Y\rangle$ be linear open. Let $v'\in\mathsf{F}(Y)^\perp$ be such that $V_{v'}\subseteq V$. By Theorem 1, $|K|\in\mathsf{F}(X)$, so $\mathcal{M}_{\mathrm{fin}}(|K|)\in\mathsf{F}(!X)$. Let $M\in V_{\mathcal{M}_{\mathrm{fin}}(|K|)\times v'}\subseteq\mathbf{k}\langle X\Rightarrow Y\rangle$, then $\widehat{M}(x)_b=0$ for each $x\in K$ and $b\in v'$. So we have $V_{\mathcal{M}_{\mathrm{fin}}(|K|)\times v'}\cap\mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle,\mathbf{k}\langle Y\rangle)\subseteq\mathcal{W}(K,V)$.

Conversely let $U\in\mathsf{F}(!X)$ and $v'\in\mathsf{F}(Y)^\perp$, then we have $u=|U|\in\mathsf{F}(X)$ and hence the subspace $K\subseteq\mathbf{k}\langle X\rangle$ of all vectors which vanish outside u is linearly compact. Let $M\in\mathbf{k}\langle X\Rightarrow Y\rangle$ be such that the map $\widehat{M}:\mathbf{k}\langle X\rangle\to\mathbf{k}\langle Y\rangle$ is polynomial and belongs to $\mathcal{W}(K,V_{v'})$. Then for any $m=[a_1,\ldots,a_n]\in\mathcal{M}_{\mathrm{fin}}(u)$ and $b\in v'$ we have $M_{m,b}=0$ because this scalar is the coefficient of the monomial $\xi_1^{m(a_1)}\ldots\xi_n^{m(a_n)}$ in the polynomial $P\in\mathbf{k}[\xi_1,\ldots,\xi_n]$ such that $P(z_1,\ldots,z_n)=\widehat{M}(x)_b$ where $x\in\mathbf{k}\langle X\rangle$ is such that $x_a=z_i$ if $a=a_i$ and $x_a=0$ if $a\notin|m|$, and $P=0$ because $\widehat{M}(K)\subseteq V_{v'}$ by assumption. Hence $M\in V_{U\times v'}$ and we have shown that $\mathcal{W}(K,V_{v'})\subseteq V_{U\times v'}\cap\mathbf{Pol}_{\mathbf{k}}(\mathbf{k}\langle X\rangle,\mathbf{k}\langle Y\rangle)$, showing that this latter set is a neighborhood of 0 in the space of polynomials. □

The Taylor formula proved in [Ehr05] for the morphisms of this Kleisli category shows that actually any morphism is the sum of a converging series whose n-th term is an homogeneous polynomial of degree n.

Remark: One should not take the world "analytic" too seriously. Our analytic maps can have unbounded degree, but when applied to an argument, for getting a scalar result, only a finite computation in the field will be needed. So they should rather be considered as some kind of generalized polynomials.

As an example, take $E=\mathbf{k}[\xi]\simeq\mathbf{k}\langle 1\Rightarrow 1\rangle$. The corresponding topology on E is the discrete topology. A typical example of analytic map is the function $d:E\to\mathbf{k}$ which maps a polynomial P to $P(P(0))$, in other words, $d(x_0+x_1\xi+\cdots+x_n\xi^n)=x_0+x_1x_0+\cdots+x_nx_0^n$. Considered as a generalized polynomial of infinitely many variables $x_0,x_1,\ldots$, we see that d is not of bounded degree. Nevertheless, it corresponds to a very simple and finite computation on polynomials.

We adress now the main topic of the paper, which is the characterization of the generalized polynomial functions in terms of extensional presheaves.

2 All quasipolynomial maps are polynomial

Definition 3 A map $f:\mathbf{k}^{\mathbf{N}}\to\mathbf{k}$ is *quasipolynomial* if, for any $n\in\mathbf{N}$ and any sequence of polynomials $\varphi_i\in\mathbf{k}[\xi_1,\ldots,\xi_n]$ $(i\in\mathbf{N})$, the map $g=\mathbf{k}^n\to\mathbf{k}$ defined by $g(x)=f(\varphi_1(x),\varphi_2(x),\ldots)$ is polynomial.

[11]It might seem more sensible to take linearly compact *affine* – instead of linear – subspaces (affine subspaces whose direction is linearly compact) in this definition, but this would not change the resulting topology because, when G is a linearly compact affine space (which is not a linear subspace), if $x\in G$ then $x\notin\mathsf{dir}\,G$ and $\mathsf{dir}\,G\oplus\mathbf{k}x$ is linearly compact (as a product of linearly compact spaces) and contains G: any linearly compact affine space is contained in a linearly compact space.

The quasipolynomial maps form a $\mathbf{k}$-algebra for the obvious operations, as easily checked.

Let f be a quasipolynomial map and let $m \in \mathcal{M}_{\mathrm{fin}}(\mathbf{N})$. We define $f_m \in \mathbf{k}$ as the coefficient of the monomial corresponding to m in the polynomial function $g : \mathbf{k}^p \to \mathbf{k}$ defined by $g(x_1, \dots, x_p) = f(x_1, \dots, x_p, 0, 0, \dots)$, for p such that the support of m is contained in $\{1, \dots, p\}$. This coefficient does not depend on the choice of p, and is well defined because f is quasipolynomial.

If $n \in \mathbf{N}$, we define $f_{|n} : \mathbf{k}^{\mathbf{N}} \to \mathbf{k}$ as the map defined by $f_{|n}(x) = f(x_1, \dots, x_n, 0, 0, \dots)$. A quasipolynomial map $f : \mathbf{k}^{\mathbf{N}} \to \mathbf{k}$ will be said to be polynomial if $f = f_{|n}$ for some n (and then of course we have $f = f_{|p}$ for all $p \geq n$). The objective of this section is to prove that (under the already mentioned assumptions on $\mathbf{k}$) any quasipolynomial map is polynomial.

Lemma 4 *Let f be quasipolynomial, and assume that $f_{|n} = 0$ for all $n \in \mathbf{N}$. Then $f = 0$. So if two quasipolynomial maps coincide on all ultimately vanishing sequences, they are equal.*

Proof. Let $x \in \mathbf{k}^{\mathbf{N}}$. Let $a_1, a_2, \dots \in \mathbf{k}$ be pairwise distinct and different from 0 (remember that $\mathbf{k}$ is assumed to be infinite). Let $g : \mathbf{k} \to \mathbf{k}$ be the map defined by

$$g(t) = f\left(x_1 \frac{a_1 - t}{a_1}, x_2 \frac{(a_1 - t)(a_2 - t)}{a_1 a_2}, \dots\right)$$

By assumption, we have $g(a_1) = g(a_2) = \dots = 0$, but g is polynomial because f is quasipolynomial, hence $g = 0$, hence $g(0) = 0$, that is, $f(x) = 0$. □

Lemma 5 *Let f be a quasipolynomial map and assume that $f_m = 0$ for almost all $m \in \mathcal{M}_{\mathrm{fin}}(\mathbf{N})$. Then one has $f(x) = \sum_{m \in \mathcal{M}_{\mathrm{fin}}(\mathbf{N})} f_m x^m$, and f is polynomial.*

This is a straightforward consequence of the above lemma. Our strategy for proving that any quasipolynomial map f is polynomial will therefore be to show that $f_m = 0$ for almost all m.

Given $i \in \mathbf{N}$, let $e_i \in \mathbf{k}^{\mathbf{N}}$ be the "i-th" canonical base vector, defined by $(e_i)_i = 1$ and $(e_i)_j = 0$ if $i \neq j$.

2.1 The partial derivatives of a quasipolynomial map

Let f be a quasipolynomial map and let $i \in \mathbf{N}$. Given $x \in \mathbf{k}^{\mathbf{N}}$, consider the map $g : \mathbf{k} \to \mathbf{k}$ defined by $g(t) = f(x + te_i)$; this function is polynomial in t, its coefficient of degree 0 is $f(x)$, and let $f'_i(x)$ be its coefficient of degree 1: we call this scalar the partial derivative of f at x wrt. its i'th parameter.

Lemma 6 *For any quasipolynomial map f, the map $f'_i : \mathbf{k}^{\mathbf{N}} \to \mathbf{k}$ is quasipolynomial.*

Proof. Let $g : \mathbf{k} \times \mathbf{k}^{\mathbf{N}} \to \mathbf{k}$ be the map defined by $g(t, x) = f(x + te_i)$. Then g is easily seen to be quasipolynomial and, by definition of f'_i, there is a map $h : \mathbf{k} \times \mathbf{k}^{\mathbf{N}} \to \mathbf{k}$ such that, for all $(t, x) \in \mathbf{k} \times \mathbf{k}^{\mathbf{N}}$, one has

$$g(t, x) = f(x) + t f'_i(x) + t^2 h(t, x) .$$

Let $n \in \mathbb{N}$ and let $\gamma : \mathbf{k}^n \to \mathbf{k}^{\mathbf{N}}$ be an $\mathbf{N}$-indexed sequence of polynomials, we must show that the map $f'_i \circ \gamma : \mathbf{k}^n \to \mathbf{k}$ is polynomial. We know that the map $P : \mathbf{k} \times \mathbf{k}^n \to \mathbf{k}$ defined by $P(t, y) = g(t, \gamma(y)) - f(\gamma(y))$ is polynomial and satisfies $P(0, y) = 0$ for all $y \in \mathbf{k}^n$, therefore, for some polynomial $Q : \mathbf{k} \times \mathbf{k}^n \to \mathbf{k}$, we have $P(t, y) = tQ(t, y)$. Hence $f'_i(\gamma(y)) = Q(t, y) - th(t, \gamma(y))$ for all $t \in \mathbf{k}$, and taking $t = 0$, we get $f'_i(\gamma(y)) = Q(0, y)$. Thus $f'_i \circ \gamma$ is polynomial. □

2.2 A family of polynomials

Given $m \in \mathcal{M}_{\mathrm{fin}}(\mathbf{N})$, let $S(m) = \sum_i m(i)i \in \mathbb{N}$. For any given $n \in \mathbb{N}$, there are only finitely many $m \in \mathcal{M}_{\mathrm{fin}}(\mathbf{N})$ such that $S(m) = n$, as easily checked. In this section, we shall make good use of this simple observation.

Let us consider the map $g : \mathbf{k} \times \mathbf{k}^{\mathbf{N}} \to \mathbf{k}$ defined by

$$g(t, x) = f(tx_1, t^2x_2, t^3x_3, \dots)\,.$$

This map is quasipolynomial since f is quasipolynomial. In particular, for any given $x \in \mathbf{k}^{\mathbf{N}}$, the map $t \to g(t, x)$ is polynomial, so that there is a family $(h_n)_{n\in\mathbf{N}}$ of functions $\mathbf{k}^{\mathbf{N}} \to \mathbf{k}$ with the following properties:

$$\forall x \in \mathbf{k}^{\mathbf{N}} \exists n \in \mathbf{N} \forall p \geq n \quad h_p(x) = 0 \tag{2}$$

and for all $x \in \mathbf{k}^{\mathbf{N}}$ and $t \in \mathbf{k}$

$$g(t, x) = \sum_{n=0}^{\infty} h_n(x)t^n\,. \tag{3}$$

By this last formula, we see that, for all $x \in \mathbf{k}^{\mathbf{N}}$ and $n \in \mathbf{N}$, we have $h_n(x) = \frac{1}{n!}g_1^{(n)}(0, x)$, and hence, by Lemma 6, each function h_n is quasipolynomial. Coming back to the definition of g, we observe that, for any $y \in \mathbf{k}^{(\mathbf{N})}$, one has, for all $n \in \mathbb{N}$: $h_n(y) = \sum_{S(m)=n} f_m y^m$, and hence, by our initial observation on S, h_n coincides with a polynomial function on all ultimately vanishing elements of $\mathbf{k}^{\mathbf{N}}$, hence h_n is polynomial by Lemma 4.

2.3 Topological considerations

Let us now make the assumption that the field $\mathbf{k}$ admits an absolute value for which it is non-discrete and complete (so for any $\varepsilon > 0$, the ball of radius ε is infinite). In this section 2.3 only, we consider $\mathbf{k}$ as equipped with the corresponding topology.

Then $\mathbf{k}^{\mathbf{N}}$ is a complete metric space, for the following distance

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, |x_n - y_n|) \tag{4}$$

which induces the product topology on $\mathbf{k}^{\mathbf{N}}$. Let $h : \mathbf{k}^{\mathbf{N}} \to \mathbf{k}$ be a polynomial map, say $h(x) = P(x_1, \dots, x_n)$ for some $P \in \mathbf{k}[\xi_1, \dots, \xi_n]$. Then since P is continuous as a function $\mathbf{k}^n \to \mathbf{k}$, we know that h is continuous, and hence $V = h^{-1}(0)$ is closed. Assume now that the interior of V is non-empty; by our assumption that $\mathbf{k}$ is non-discrete, any open ball in $\mathbf{k}$ is infinite, so we can find an infinite subset I of $\mathbf{k}$ such that P vanishes on I^n and hence $P = 0$, therefore $h = 0$, that is, $V = \mathbf{k}^{\mathbf{N}}$.

Let $V_n \subset \mathbf{k}^{\mathbf{N}}$ be the set of all xs such that $h_p(x) = 0$ for all $p \geq n$. Then $(V_n)_{n\in\mathbb{N}}$ is an increasing family of closed subsets of $\mathbf{k}^{\mathbf{N}}$ whose union is $\mathbf{k}^{\mathbf{N}}$ by Property (2). Therefore, by the Baire categoricity theorem (it applies to $\mathbf{k}^{\mathbf{N}}$ since this space is a complete metric space), there must exist some n such that V_n has a non-empty interior. But then, for each $p \geq n$, ${h_n}^{-1}(0)$ must have a non-empty interior since indeed $V_n = \bigcap_{p\geq n} {h_p}^{-1}(0)$, and hence we must have $h_p = 0$ for all $p \geq n$. This proves that g is polynomial, and hence f is polynomial, since $f(x) = g(1, x)$.

To summarize, we have proved the following result.

Theorem 7 *Assume that* $\mathbf{k}$ *admits an absolute value for which it is non-discrete and complete. Then any quasipolynomial function from* $\mathbf{k}^{\mathbf{N}}$ *to* $\mathbf{k}$ *is polynomial.*

We generalize this result to finiteness spaces.

2.4 Quasipolynomial maps on finiteness spaces

Given $n \in \mathbb{N}$, let $\underline{n}$ be the finiteness space whose web is $\{1, \dots, n\}$, so that $\mathbf{k}\langle\underline{n}\rangle = \mathbf{k}^n$, with the discrete topology.

Let X and Y be finiteness spaces. A quasipolynomial map f from X to Y is a function $f : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$ with the following property: for any $n \in \mathbb{N}$ and any $\gamma \in \mathbf{k}\langle \underline{n} \Rightarrow X\rangle$, the map $f \circ \gamma : \mathbf{k}\langle\underline{n}\rangle \to \mathbf{k}\langle Y\rangle$ belongs to $\mathbf{k}\langle \underline{n} \Rightarrow Y\rangle$. Obviously, any $f \in \mathbf{k}\langle X \Rightarrow Y\rangle$ is a quasipolynomial map from X to Y.

Theorem 8 *If $\mathbf{k}$ admits an absolute value for which it is non-discrete and complete, then any quasipolynomial map from X to Y is an element of $\mathbf{k}\langle X \Rightarrow Y\rangle$.*

Proof. Let $f : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$ be quasipolynomial. Let $b \in |Y|$ and let $f^b : \mathbf{k}\langle X\rangle \to \mathbf{k}$ be f composed with the projection $\mathbf{k}\langle Y\rangle \to \mathbf{k}$ which maps y to y_b (it is a continuous linear form on $\mathbf{k}\langle Y\rangle$). Given $u \in \mathsf{F}(X)$, we define $j_u : \mathbf{k}^u \to \mathbf{k}\langle X\rangle$ as the canically associated inclusion (any element $x \in \mathbf{k}^u$ is mapped by j_u to x extended with 0's for the indices which do not belong to u).

Let $m \in |!X|$ and let u be any finite subset of $|X|$ such that $|m| \subseteq u$. We denote by f^b_m the coefficient of the monomial m in the polynomial function $f \circ j_u$; this coefficient does not depend on the choice of u (for being more precise, let $a_1, \dots, a_n$ be a repetition-free enumeration of $|m|$, then f^b_m is the coefficient of the monomial $\xi_1^{m(1)} \cdots \xi_n^{m(n)}$ in the polynomial $P \in \mathbf{k}[\xi_1, \dots, \xi_n]$ such that $f^b(x) = P(x_{a_1}, \dots, x_{a_n})$ for any $x \in \mathbf{k}\langle X\rangle$ which vanishes outside $|m|$). Let $R = \{(m, b) \mid f^b_m \neq 0\} \subseteq |X \Rightarrow Y| = |!X| \times |Y|$, we show that $R \in \mathsf{F}(X \Rightarrow Y)$

Let $u \in \mathsf{F}(X)$. Fix first some $b \in |Y|$. If u is finite, our quasipolynomiality assumption implies directly that $f \circ j_u$ is polynomial. Otherwise, our assumption implies that $f \circ j_u$ is quasipolynomial in the sense of Section 2, and hence is polynomial by Theorem 7. Therefore, there are only finitely many ms such that $(m, b) \in R$ and $|m| \subseteq u$.

To conclude that $R \in \mathsf{F}(X \Rightarrow Y)$, we must show that the set $R(u^!) = \{b \mid \exists m\ (m, b) \in R \text{ and } |m| \subseteq u\}$ belongs to $\mathsf{F}(Y)$. So assume that this set is infinite and let $b_1, b_2, \dots$ be a repetition-free enumeration thereof. Let $f_i : \mathbf{k}^u \to \mathbf{k}$ be the restriction of f^{b_i} to $\mathbf{k}^u$. Each f_i is polynomial.

We show that there exists $x \in \mathbf{k}^u$ such that $f_i(x) \neq 0$ for each i. Assume this is not the case and let $g_n = f_1 \cdots f_n$ for each $n \in \mathbb{N}$. Then each $g_n : \mathbf{k}^u \to \mathbf{k}$ is a polynomial function. For each $x \in \mathbf{k}^u$, there exists i such that $f_i(x) = 0$ and hence there exists n such that $g_p(x) = 0$ for each $p \geq n$. Let $V_n = {g_n}^{-1}(0)$.

As at the end of the proof of Theorem 7, we see that each V_n is a closed subset of $\mathbf{k}^u$ (for the product topology), and that V_n is an increasing family with $\bigcup_{n \in \mathbb{N}} V_n = \mathbf{k}^{\mathbf{N}}$, and therefore that there exists n such that $V_n = \mathbf{k}^{\mathbf{N}}$ by Baire theorem. Then we have $g_n = 0$, but $f_1, \dots, f_n$ can be seen as the elements of a common algebra of polynomials $\mathbf{k}[\xi_1, \dots, \xi_N]$ (for N sufficiently large), and we have $f_1 \cdots f_n = 0$. An algebra of polynomials over a field is an integral domain, and hence there is i such that $f_i = 0$. But this is impossible, since we know that there is m such that $|m| \subseteq u$ and $(m, b_i) \in R$. So there is $x \in \mathbf{k}^u$ such that $f_i(x) \neq 0$ for each i, that is $f(j_u(x))_b \neq 0$ for each $b \in R(u^!)$. Since $f(j_u(x)) \in \mathbf{k}\langle Y\rangle$, we conclude that $R(u^!) \in \mathsf{F}(Y)$ as announced.

To conclude with the proof of the theorem, we must show that, for each $b \in |Y|$, and each $x \in \mathbf{k}\langle x\rangle$, one has

$$f(x)_b = \sum_{m \in |!X|} f^b_m x^m ,$$

we know that the right hand sum is finite because $R \in \mathsf{F}(X \Rightarrow Y)$. For proving the equation, it suffices to observe that this equation holds for xs which have a finite support (by the very definition of the coefficients f^b_m) and to apply once again Lemma 4. □

The remainder of the paper is essentially a categorical reformulation of this result.

3 A functor from finiteness spaces to extensional presheaves

Given any locally small category $\mathcal{C}$, the Yoneda embedding $\mathcal{Y} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the functor which maps any object C of $\mathcal{C}$ to $\mathcal{C}(\_, C)$. More generally, given a functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is another locally small category, we can define a functor $\mathcal{Y}_F : \mathcal{D} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ which maps D to $\mathcal{D}(F(\_), D)$.

3.1 The extensional presheaves on $\mathbf{Pol}_\mathbf{k}$.

We consider the particular case where $\mathcal{C} = \mathbf{Pol}_\mathbf{k}$, the Lawvere theory of polynomials with coefficients in $\mathbf{k}$. Among all the objects of $\mathbf{Set}^{\mathbf{Pol}_\mathbf{k}^{\mathrm{op}}}$, we distinguish the *extensional presheaves* [RS99, Str05], of which we give now a direct definition.

An extensional presheaf on $\mathbf{Pol}_\mathbf{k}$ is a pair $A = (\mathsf{S}(A), \mathsf{R}(A))$ where $\mathsf{S}(A)$ is a set and $\mathsf{R}(A) = (\mathsf{R}(A)_p)_{p\in\mathbb{N}}$ is a family of sets satisfying the following conditions:

- $\mathsf{R}(A)_p \subseteq \mathsf{S}(A)^{\mathbf{k}^p}$ and $\mathsf{R}(A)_p$ contains all the constant functions from $\mathbf{k}^p$ to $\mathsf{S}(A)$ (this latter condition corresponds to the extensionality of the corresponding presheaf).
- If $\gamma \in \mathsf{R}(A)_q$ and $f \in \mathbf{Pol}_\mathbf{k}(p, q)$, then $\gamma \circ f \in \mathsf{R}(A)_p$.

Observe in particular that $\mathsf{S}(A) = \mathsf{R}(A)_0$.

Given two extensional presheaves A and B, a morphism from A to B is a function $h : \mathsf{S}(A) \to \mathsf{S}(B)$ such that, for all p and all $\gamma \in \mathsf{R}(A)_p$, one has $h \circ \gamma \in \mathsf{R}(B)_p$.

An extensional presheaf A is of course a presheaf, also denoted as A: take $A_n = \mathsf{R}(A)_n$ and, if $f \in \mathbf{Pol}_\mathbf{k}(n, p)$, define $A_f : A_p \to A_n$ by precomposition. More interesting is the simple observation that a morphism from A to B (as extensional presheaves) is just the same as a natural transformation $t : A \to B$ (considered as presheaves): the function which maps t to the morphism of extensional presheaves t_0 is injective. So we consider the category of extensional presheaves as a full subcategory of $\mathbf{Set}^{\mathbf{Pol}_\mathbf{k}^{\mathrm{op}}}$. If $n \in \mathbb{N}$, observe that $\mathcal{Y}n$ is an extensional presheaf.

We sketch the proof that the category of extensional presheaves is a full sub-cartesian closed category of that of general presheaves, and describe the corresponding constructions.

Given A and B two extensional presheaves, their cartesian product is $A \times B$, where $\mathsf{S}(A \times B) = \mathsf{S}(A) \times \mathsf{S}(B)$. An element γ of $\mathsf{S}(A \times B)^{\mathbf{k}^p}$ belongs to $\mathsf{R}(A \times B)_p$ iff $\pi_1 \circ \gamma \in \mathsf{R}(A)_p$ and $\pi_2 \circ \gamma \in \mathsf{R}(B)_p$. The projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ are defined in the obvious way. This is just the same operation as the product of A and B considered as presheaves.

The function space $A \Rightarrow B$ is defined as follows. The set $\mathsf{S}(A \Rightarrow B)$ is the collection of all extensional presheaves morphisms from A to B. The set $\mathsf{R}(A \Rightarrow B)_p$ is the collection of all $h : \mathbf{k}^p \to \mathsf{S}(A \Rightarrow B)$ such that, for any $q \in \mathbb{N}$ and any $\gamma \in \mathsf{R}(A)_q$, the mapping $\gamma' : \mathbf{k}^p \times \mathbf{k}^q \simeq \mathbf{k}^{q+p} \to \mathsf{S}(B)$ given by $\gamma'(x, y) = h(x)(\gamma(y))$ belongs to $\mathsf{R}(B)_{p+q}$. Evaluation $\mathrm{ev} : (A \Rightarrow B) \times A \to B$ is defined in the usual way, as in the category of sets and functions, and similarly for the curryfication $h : C \to A \Rightarrow B$ of a morphism $g : C \times A \to B$.

Consider now A and B as presheaves. Their function space in the category of presheaves is given by $(A \Rightarrow B)_p = \mathbf{Set}^{\mathbf{Pol}_\mathbf{k}^{\mathrm{op}}}(\mathcal{Y}p \times A, B)$, but both $\mathcal{Y}p \times A$ and B are extensional presheaves, and so an element of that homset is just a function $h : k^p \times A_0 \to B_0$ which is a morphism of extensional presheaves from $\mathcal{Y}p \times A$ to B. Spelling out the definition (and using the fact that $\mathbf{Pol}_\mathbf{k}$ is cartesian) one checks that such functions are in bijective correspondence with the elements of $\mathsf{R}(A \Rightarrow B)_p$. Again, $A \Rightarrow B$ is isomorphic to the function space of A and B, considered as presheaves.

3.2 Extending the Yoneda embedding

Observe that, if $n, p \in \mathbb{N}$, then $\mathbf{k}\langle \underline{n} \Rightarrow \underline{p} \rangle = \mathbf{Pol}_\mathbf{k}(n, p)$ and this identification is compatible with composition, in the sense that it defines an isomorphism between $\mathbf{Pol}_\mathbf{k}$ and the full subcategory of $\mathbf{Fin}_\mathbf{k}^!$ whose objects are the $\underline{n}$'s. Let $J : \mathbf{Pol}_\mathbf{k} \to \mathbf{Fin}_\mathbf{k}$ be the corresponding inclusion functor.

So we have a functor $\mathcal{Y}_J : \mathbf{Fin}_\mathbf{k} \to$ to $\mathbf{Set}^{\mathbf{Pol}_\mathbf{k}^{\mathrm{op}}}$ which extends $\mathcal{Y}$ (in the sense that $\mathcal{Y}_J \circ J = \mathcal{Y}$) and whose range lies in the full subcategory of extensional presheaves, as easily checked. Explicitly, $\mathsf{S}(\mathcal{Y}_J X) = \mathbf{k}\langle X \rangle$

and $\mathsf{R}(\mathcal{Y}_J X)_n = \mathbf{Fin}^!_{\mathbf{k}}(\underline{n}, X)$ (a space of $\mathbf{k}\langle X\rangle$-valued polynomial functions). Now we can state and prove the main result of the paper.

Theorem 9 *Assume that* $\mathbf{k}$ *admits an absolute value for which it is non-discrete and complete. Then the functor* $\mathcal{Y}_J$ *is full and faithful and preserves the cartesian closed structure of* $\mathbf{Fin}^!_{\mathbf{k}}$.

Proof. Faithfulness is trivial. Let us check fullness. Let $f : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$ be an extensional presheaf morphism from $\mathcal{Y}_J X$ to $\mathcal{Y}_J Y$. This means that f is quasipolynomial in the sense of Section 2.4 and hence $f \in \mathbf{Fin}^!_{\mathbf{k}}(X, Y)$ by Theorem 8.

The functor $\mathcal{Y}_J$ preserves products (it preserves all existing limits) by definition, so we just have to prove that it preserves function spaces, that is, we have to exhibit a natural isomorphism between the extensional presheaves $\mathcal{Y}_J(X \Rightarrow Y)$ and $\mathcal{Y}_J X \Rightarrow \mathcal{Y}_J Y$. We have $\mathsf{R}(\mathcal{Y}_J(X \Rightarrow Y))_n = \mathbf{Fin}^!_{\mathbf{k}}(\underline{n}, X \Rightarrow Y) \simeq \mathbf{Fin}^!_{\mathbf{k}}(\underline{n} \,\&\, X, Y)$. But by Theorem 8 again, $\mathbf{Fin}^!_{\mathbf{k}}(\underline{n} \,\&\, X, Y) \simeq \mathbf{Set}^{\mathbf{Pol}^{\mathrm{op}}_{\mathbf{k}}}(\mathcal{Y}_J(\underline{n} \,\&\, X), \mathcal{Y}_J Y) \simeq \mathbf{Set}^{\mathbf{Pol}^{\mathrm{op}}_{\mathbf{k}}}(\mathcal{Y}_J \underline{n} \times \mathcal{Y}_J X, \mathcal{Y}_J Y) = \mathbf{Set}^{\mathbf{Pol}^{\mathrm{op}}_{\mathbf{k}}}(\mathcal{Y} n \times \mathcal{Y}_J X, \mathcal{Y}_J Y) = (\mathcal{Y}_J X \Rightarrow \mathcal{Y}_J Y)_n$ by definition of function spaces in $\mathbf{Set}^{\mathbf{Pol}^{\mathrm{op}}_{\mathbf{k}}}$. The naturality of these isomorphisms is easily checked. $\square$

4 Conclusion

The hypothesis that the field should admit an absolute value for which it is complete and non-discrete is rather unnatural since, in the category of Lefschetz spaces, the field is taken with the discrete topology. The next step, to turn the main result of the paper into a purely algebraic one, will be to get rid of this hypothesis, if possible.

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